MATH 521A: Abstract Algebra

Preparation for Final Exam

R, S are rings, not necessarily commutative or with identity

F is a field.

1. Carefully define the terms gcd, ring, quotient ring, integral domain, field, $F[x]$, $\mathbb{Z}_n$, irreducible element, kernel, image, prime element, ideal, maximal ideal, prime ideal, minimal polynomial, dimension (of a field extension).

2. Carefully the state the following theorems: division algorithm in $\mathbb{Z}$, division algorithm in $F[x]$, fundamental theorem of arithmetic, remainder theorem, Gauss's lemma, rational root test, Eisenstein's criterion, first isomorphism theorem (book's version or my version).

3. Let $a, b, c, d \in \mathbb{Z}$, and $n \in \mathbb{N}$. Suppose that $a \equiv b \pmod{n}$ and $c \equiv d \pmod{n}$. Prove that $ac \equiv bd \pmod{n}$.

4. Let $a, b \in \mathbb{Z}$. Prove that $\gcd(a, b) | \gcd(a + b, a - b)$, assuming that both numbers exist.

5. Let $p \in \mathbb{N}$ be irreducible. Prove that $p^4 + 14$ is reducible.

6. We call $r \in R$ *idempotent* if $r^2 = r$. Suppose that R has 1, and let $x \in R$ be idempotent. Prove that $1 - x$ is idempotent.

7. Let $f : R \to S$ be a ring isomorphism. Prove that R has an identity, if and only if, S has an identity.

8. Let F be a field, and let $a, b \in F$. Prove that $\gcd(x^2 + a, x + b) = 1$ in $F[x]$, if and only if $a \neq -b^2$.

9. Find the equivalence classes and rules for addition and multiplication in $\mathbb{Q}[x]/(x^2 - 9)$. Find all the units and zero divisors.

10. Let $f(x), g(x), h(x) \in F[x]$. Suppose that $\gcd(f(x), g(x)) = 1$ and that $f(x)|g(x)h(x)$. Prove that $f(x)|h(x)$.

11. Let $f(x), g(x), h(x), p(x) \in F[x]$, with $p(x) \neq 0$. Prove that $f(x)h(x) \equiv g(x)h(x) \pmod{p(x)}$, if and only if, $f(x) \equiv g(x) \pmod{\frac{p(x)}{\gcd(h(x),p(x))}}$.

12. Let $f(x), g(x), h(x), k(x), p(x) \in F[x]$. Suppose that $f(x) \equiv g(x) \pmod{p(x)}$ and $h(x) \equiv k(x) \pmod{p(x)}$. Prove that $f(x)h(x) \equiv g(x)k(x) \pmod{p(x)}$.

13. Prove that (n) is a prime ideal of $\mathbb{Z}$, if and only if, n is either prime or zero.
14. Find a ring homomorphism $f : \mathbb{Z} \to \mathbb{Z}[x]$, such that the image of f is not an ideal.
15. Let $a \in F$ and define $\phi_a : F[x] \to F$ via $\phi_a : f(x) \mapsto f(a)$. Prove that ϕ_a is a surjective ring homomorphism.
16. Let $a \in F$ and define $\phi_a : F[x] \to F$ via $\phi_a : f(x) \mapsto f(a)$. Compute the kernel of ϕ_a. What does the First Isomorphism Theorem tell you here?
17. Define $I \subseteq \mathbb{Z}_3[x]$ via $I = \{f(x) : f(0)f(1) = 0\}$. Prove or disprove that I is an ideal in $\mathbb{Z}_3[x]$.
18. Prove that the principal ideal $(x - 1)$ in $\mathbb{Z}[x]$ is prime but not maximal.
19. Find the minimal polynomial of $\sqrt{5 + \sqrt{8}}$ over $\mathbb{Q}$.
20. Find the minimal polynomial of $\sqrt{3 + \sqrt{8}}$ over $\mathbb{Q}$.
    Hint: the answer is of a different degree than for the previous problem.